Exercise 33

A particle moves in a velocity field $\mathbf{V}(x, y) = \langle x^2, x + y^2 \rangle$. If it is at position $(2, 1)$ at time $t = 3$, estimate its location at time $t = 3.01$.

Solution

The relationship between velocity and position in one dimension is

$$v = \frac{dx}{dt}.$$

For the vectors here, it becomes

$$\mathbf{V} = \frac{d\mathbf{X}}{dt}.$$

We don't know what the position is, so we'll approximate the derivative by the difference quotient.

$$\mathbf{V} \approx \frac{\mathbf{X}(t) - \mathbf{X}(t_0)}{t - t_0}$$

Evaluate the velocity at the particle's initial position, $x = 2$ and $y = 1$, and plug in $t = 3.01$ and $t_0 = 3$ on the right side.

$$\mathbf{V}\Big|_{\substack{x=2 \\ y=1}} \approx \frac{\mathbf{X}(3.01) - \mathbf{X}(3)}{3.01 - 3}$$

$$\langle 4, 3 \rangle \approx \frac{\mathbf{X}(3.01) - \langle 2, 1 \rangle}{0.01}$$

Solve for $\mathbf{X}(3.01)$, the position vector of the particle at $t = 3.01$.

$$\langle 0.04, 0.03 \rangle + \langle 2, 1 \rangle \approx \mathbf{X}(3.01)$$

Therefore,

$$\mathbf{X}(3.01) \approx \langle 2.04, 1.03 \rangle.$$